COMBINATORICA 6 (4) (1986) 299—308

ADJOINTS OF ORIENTED MATROIDS

A. BACHEM and W. KERN

Received 6 March 1984

An adjoint of a geometric lattice G is a geometric lattice $\tilde{G}$ of the same rank into which there is an embedding e mapping the copoints of G onto the points of $\tilde{G}$. In this paper we introduce oriented adjoints and prove that they can be embedded into the extension lattice of oriented matroids.

1. Introduction

As shown by Billera and Munson [1], the usual polarity properties of the face lattices of convex polytopes do not extend to oriented matroids. More specifically the upside down lattice of a face lattice of an oriented matroid M is in general not again a face lattice of some other oriented matroid M^*. Thus oriented matroids do not always have a polar. The reason is that existence of polars would imply the existence of adjoints and as Chéung [5] showed the Vámos matroid does not have an adjoint.

We discuss adjoints and introduce the notion of oriented adjoints in section 2. In section 3 we show that the elements of the signed cocircuit span of an oriented adjoint represent point extensions of the original oriented matroid, i.e. the oriented adjoint of an orient matroid M can be embedded into the extension lattice of M. This result can be used to prove that the existence of oriented adjoints is sufficient for the existence of polars.

We shall use the notation of Las Vergnas ([9]) with some minor exceptions. We call a $(+,-,0)$-vector $X=(X_e)_{e\in E}\in\{+,-,0\}^E$ a signed vector and often identify X with the pair (X_+, X_-) where $X_{+(-)}:=\{e\in E|X_e=+(-)\}$. As usual $\underline{X}=X_+\cup X_-$ denotes the set underlying X and $X_0:=E\setminus\underline{X}$ its complement. An *oriented matroid* then is the collection $\mathcal{O}$ of signed sets on $E=\{1, \dots, n\}$ satisfying

(1) $(\emptyset,\emptyset)\in\mathcal{O}$, $X\in\mathcal{O}$ implies $-X\in\mathcal{O}$.
(2) $X, Y\in\mathcal{O}$ imply (the composition) $XY\in\mathcal{O}$.
(3) If $X, Y\in\mathcal{O}$ and $e\in E$ separates X and Y, then there exists $Z\in\mathcal{O}$ such that $Z_e=0$ and $Z_f=(X, Y)_f$ for every $f\in E$ that does not separate X and Y.

For an oriented matroid $\mathcal{O}$, $\mathscr{C}(\mathcal{O})$ denotes the set of circuits, and $L:=\{X_0|X\in\mathcal{O}\}$ is the lattice of flats of the matroid $\underline{\mathcal{O}}$ underlying the oriented matroid $\mathcal{O}$.

Supported by Sonderforschungbereich 21 (DFG).
AMS subject classification (1980): 05 B 35

2. Adjoints of oriented matroids

An *adjoint* of a geometric lattice G is a geometric lattice $\tilde{G}$ of the same rank into which there is an embedding (i.e. a one-one order-reversing function) $e: G \to \tilde{G}$, mapping the copoints of G onto the points of $\tilde{G}$. Similarily we call a matroid $\tilde{M}$ an adjoint of a matroid M if the lattice of flats of $\tilde{M}$ is an adjoint of the lattice of flats of M. Clearly e^{-1} restricted to points of $\tilde{G}$ defines a bijection φ between points of $\tilde{G}$ and copoints of G. This map φ shows how the embedding works. Take any flat F of the matroid M and consider all hyperplanes $H_i\,(i\in I)$ of M which are supersets of F. Then $\varphi^{-1}(H_i)$ $(i\in I)$ are points in $\tilde{G}$ with join $\tilde{F}=\bigcup_{i\in I}\varphi^{-1}(H_i)$ and indeed $\tilde{F}=e(F)$.

Conversely, assume we are given the mapping φ. What kind of condition do we have to add in order to establish such an embedding e? Clearly, we need to guarantee that e is one-to-one and we shall see that it suffices to assume the existence of an injective map ψ mapping the points of M into the copoints of $\tilde{M}$ such that for all points p of M and $\tilde{p}\in\tilde{M}$, $p\leqq\varphi(\tilde{p})$ if and only if $\tilde{p}\leqq\psi(p)$ holds. Indeed assume $(H_i)_{i\in I}$ and $(H_i')_{i\in I'}$, are all hyperplanes of M containing the points p respec. q of M. Then $e(p)$ respec. $e(q)$ equals the join of $(\varphi^{-1}(H_i))_{i\in I}$ repec. $(\varphi^{-1}(H_i'))_{i\in I'}$ in $\tilde{M}$. And since $p\leqq\varphi(\varphi^{-1}(H_i))$ we have $\varphi^{-1}(H_i)\leqq\psi(p)$. Thus $e(p)=e(q)$ implies $\psi(p)=\psi(q)$ and hence $p=q$. Thus we have:

Proposition 2.1. *Let M and $\tilde{M}$ be two matroids of the same rank. Then $\tilde{M}$ is an adjoint of M if and only if there is a bijective map φ from the points of $\tilde{M}$ onto the hyperplanes of M and a one-to-one map ψ from the points of M into the hyperplanes of $\tilde{M}$ such that*

$$p\leqq\varphi(q) \text{ if and only if } q\leqq\psi(p) \tag{2.1}$$

holds for all points p and q of M respec. $\tilde{M}$. ∎

Example. Let $E\subset\mathbf{R}^n$ finite and suppose that E has dimension n. Let L denote the lattice of subspaces spanned by subsets of E. Since $\dim(E)=n$, the copoints of E correspond to hyperplanes of $\mathbf{R}^n$. For every copoint of L choose one of its normal vectors. Then the lattice L^{Δ}, consisting of all subspaces spanned by subsets of E^{Δ} is easily seen to be an adjoint of L.

This shows how adjoints may be constructed using the orthogonalily relations in linear vectorspaces. If a matroid is representable over different fields, the above construction may be carried out in each of them, sometimes yielding different adjoints (cf. Bixby and Coullard [2]). Thus adjoints are not unique. They also may even fail to exist (cf. Chéung [5]).

The reformulation, given in Proposition 2.1 above, of Chéung's notion ([5]) of an adjoint of a geometry generalizes very naturally to oriented matroids.

Definition. Let $\mathcal{O}$ and $\tilde{\mathcal{O}}$ be two oriented matroids without loops and of the same rank. Then $\tilde{\mathcal{O}}$ is called an *adjoint* of $\mathcal{O}$ if there are maps

$$\varphi:\tilde{E}\to\mathscr{C}(\mathcal{O}^{\perp}) \quad\text{and}\quad \psi:E\to\mathscr{C}(\tilde{\mathcal{O}}^{\perp}) \tag{2.2}$$

such that for all elements $e \in E$ and $\tilde{e} \in \tilde{E}$

(2.3) $e \in \varphi(\tilde{e})_\pm$ if and only if $\tilde{e} \in \psi(e)_\pm$ holds and the induced map $\underline{\varphi}$ mapping points of $\tilde{\mathcal{O}}$ onto hyperplanes $H = E \setminus \underline{\varphi(\tilde{e})}$ $(\tilde{e} \in \tilde{E})$ of $\mathcal{O}$ is bijective.

The following lemma shows that the induced map $\underline{\varphi}$ is well defined.

Lemma 2.2. *If $\tilde{p}$ is a point of $\tilde{\mathcal{O}}$ and $\tilde{e}_1, \tilde{e}_2 \in \tilde{p}$, then $\varphi(\tilde{e}_1) = \pm\varphi(\tilde{e}_2)$.*

Proof. Clearly if $\tilde{e}_1, \tilde{e}_2 \in \tilde{p}$ then there is a circuit $\tilde{C}$ of $\tilde{\mathcal{O}}$ containing only $\tilde{e}_1$ and $\tilde{e}_2$ and up to opposite signs either $\tilde{e}_1, \tilde{e}_2 \in \tilde{C}_+$ or $e_1 \in \tilde{C}_+$ and $\tilde{e}_2 \in \tilde{C}_-$. If $\tilde{e}_1, \tilde{e}_2 \in \tilde{C}_+$ the orthogonality property $(\tilde{C}_+ \cap \tilde{C}^\perp_\pm) \cup (\tilde{C}_- \cap \tilde{C}^\perp_\mp) \neq \emptyset$ for any cocircuit $\tilde{C}^\perp$ containing $\tilde{e}_1$ implies $\tilde{e}_1 \in \underline{\tilde{C}^\perp}$ if and only if $\tilde{e}_2 \in \underline{\tilde{C}^\perp}$. Thus choosing $\tilde{C}^\perp = \psi(e)$ for any $e \in E$ we obtain $\tilde{e}_1 \in \psi(e)_+$ if and only if $\tilde{e}_2 \in \psi(e)_+$. By (2.3) this proves $e \in \varphi(\tilde{e}_1)_+$ if and only if $e \in \varphi(\tilde{e}_2)_+$ for all $e \in E$. Hence $\varphi(\tilde{e}_1) = \varphi(\tilde{e}_2)$. Similarily, $\tilde{e}_1 \in \tilde{C}_+$ and $\tilde{e}_2 \in \tilde{C}_-$ gives $\varphi(\tilde{e}_1) = -\varphi(\tilde{e}_2)$. ∎

In the same way as above we may also define the induced map $\underline{\psi}$ mapping points of E into hyperplanes of $\tilde{\mathcal{O}}$. By the same arguments $\underline{\psi}$ is also well defined.

Proposition 2.3. *If $\tilde{\mathcal{O}}$ is an adjoint of the oriented matroid $\mathcal{O}$, then $\underline{\tilde{\mathcal{O}}}$ is an adjoint of $\underline{\mathcal{O}}$.*

Proof. The map $\underline{\varphi}$ is bijective by definition. Let us show that $\underline{\psi}$ is one-to one, i.e. assume $\underline{\psi}(p_1) = \underline{\psi}(p_2)$ for two points p_1, p_2 of $\mathcal{O}$. Let H be any hyperplane of $\mathcal{O}$. Then by (2.3) $p_1 \leqq H = \underline{\varphi}(\tilde{q})$ (for some point $\tilde{q}$) if and only if $\tilde{q} \leqq \underline{\psi}(p_1) = \underline{\psi}(p_2)$ and again by (2.3) if and only if $p_2 \leqq H = \underline{\varphi}(\tilde{q})$. This implies $p_1 = p_2$, thus $\underline{\psi}$ is injective. Since (2.1) follows from (2.3) this proves the proposition. ∎

We think it is clear from the example given above, how to construct adjoints of oriented matroids which are, say, representable over the reals; there again, adjoints are not uniquely determined. As an example, consider the oriented matroid of rank three, corresponding to six points in the plane forming a regular hexagon.

If we construct an adjoint, the three points of the adjoint which correspond to the three diagonals are on a line, since the three diagonals are concurrent. If, however, any of the vertices of the hexagon is slightly perturbed, the construction will yield a different adjoint, though the oriented matroid on the given set of six points does not change. It is clear also from this example that an oriented matroid in general is not uniquely determined by its adjoint: Whether or not the common point of intersection of the three diagonals is in the matroid does not affect the adjoint.

As far as existence of adjoints is concerned, the situation is still worse in the oriented case. As for unoriented matroids, it is not hard to prove that the class of oriented matroids which do have an adjoint is closed under minors. In [10], a finite number of minor minimal nonrepresentable oriented matroids has been given. Mandel has all of them proved to be "non-euclidean" and hence they cannot have an adjoint (cf. section 4).

As far as rank three is concerned, the situation is similar as for unoriented matroids: Every rank three oriented matroid does have an adjoint (cf. [6]).

3. Embedding adjoints into the extension lattice

It is well known [8] that every point extension $N=(E\cup p, \mathscr{C}')$ of a matroid $M=(E, \mathscr{C})$ corresponds uniquely to a "linearily closed" subset $\mathscr{L}$ of the set of hyperplanes of M. Such a subset is usually called a "localization" or a "linear subclass". The set of all localizations of M ordered by inclusion forms a lattice $E(M)$, called the *extension lattice* of M.

If $\tilde{M}$ is an adjoint of M then by definition the points of $\tilde{M}$ correspond to hyperplanes of M. Hence all points beneath a given flat $\tilde{F}$ of $\tilde{M}$ define a localization $L_{\tilde{F}}$ of M. Cheung ([5]) proved that this mapping $f=\tilde{F}\to L_{\tilde{F}}$ is an embedding of the geometric lattice of flats of $\tilde{M}$ into the extension lattice $E(M)$ of M.

This implies that whenever M has an adjoint $\tilde{M}$, then we can "intersect" flats of M by adding a new point. More precisely, we say that we can intersect two flats F, G of M, if a) (F, G) is a nonmodular pair, i.e. $r_M(F\cap G)+r_M(F\cup G)<r_M(F)+r_M(G)$ and b) there exists a point extension M' of M such that $r_{M'}(\mathrm{cl}_{M'}(F)\cap\mathrm{cl}_{M'}(G))>r_M(F\cap G)$.

Proposition 3.1. *If M has an adjoint $\tilde{M}$, then any two flats F, G of M can be intersected.*

Proof. Let L and $\tilde{L}$ be the lattice of flats of M and $\tilde{M}$ resp., let $e: L\to\tilde{L}$ be the embedding and let $f:\tilde{L}\to E(M)$ be the map that assigns to every $\tilde{F}\in\tilde{L}$ the corresponding localization $L_{\tilde{F}}=\{H\mid H$ is a copoint of M and $e(H)\in\tilde{F}\}$. If $F\in L$, then $\tilde{F}:=e(F)\in\tilde{L}$ and $f(\tilde{F})$ is obviously the localization of M that corresponds to a "principal extension" of M on F, i.e. $f(\tilde{F})=\{H\mid H$ is a hyperplane of M and $H\geqq F\}$. Now let (F, G) be a nonmodular pair of flats of M and let $\tilde{F}=e(F)$ and $\tilde{G}=e(G)$. In order to show that F and G can be intersected, we consider the extension $M'=M\cup p$ defined by the localization $f(\tilde{F}\vee\tilde{G})$. Since e is an anti-isomorphism, $\tilde{r}(\tilde{F})=r(M)-r(F)$, where $\tilde{r}$ is the rank function in $\tilde{M}$. Moreover, we have $e(F\vee G)=e(F)\wedge e(G)=\tilde{F}\cap\tilde{G}$. Thus $\tilde{r}(\tilde{F}\vee\tilde{G})\leqq\tilde{r}(\tilde{F})+\tilde{r}(\tilde{G})-\tilde{r}(\tilde{F}\cap\tilde{G})=r(M)-r(F)+r(M)-r(G)-(r(M)-r(F\vee G))<r(M)-r(F\cap G)=\tilde{r}(e(F\cap G))$, since (F, G) is nonmodular. Hence $\tilde{F}\vee\tilde{G}\subsetneqq e(F\cap G)$ and thus $\mathscr{L}:=f(H)=(\tilde{F}\vee\tilde{G})$ is a localization of M which is strictly contained in the localization of M which corresponds to the principal extension of M on $F\cap G$. Thus the extension $M'=M\cup p$ determined by $\mathscr{L}$ is such that the "new point" p lies in the closure of F and G but not in the closure of $F\cap G$. This proves the proposition. ∎

Proposition 3.1 can be used in order to show that adjoints do not always exist, see [5].

For an oriented matroid $\mathcal{O}$ a pair $(\mathscr{Y}, \mathscr{X})$ of collections of cocircuits of $\mathcal{O}$ is a *localization* of M if $\mathscr{Y}\cup-\mathscr{Y}\cup\mathscr{X}$ is a partition of the cocircuits of $\mathcal{O}$, and $\mathscr{C}:=\mathscr{Y}\cup\mathscr{X}$ satisfies the elimination axiom ([10]). Again Las Vergnas ([8]) proved that there exists a bijection between the set of all point extensions of $\mathcal{O}$ and the set of all localizations of $\mathcal{O}$. Using the relation $(\mathscr{Y}, \mathscr{X})\leqq(\mathscr{Y}', \mathscr{X}')$ if and only if $\mathscr{Y}\subseteqq\mathscr{Y}'$ we shall now consider the partially ordered set $E(\mathcal{O})$ of all localizations of $\mathcal{O}$ and call it the *extension lattice* of $\mathcal{O}$. (This notion is used in analogy to the unoriented case, though here $E(\mathcal{O})$ is *not* a lattice in general. Indeed, every extension of $\mathcal{O}$ by a point "in general position" corresponds to a maximal element in $E(\mathcal{O})$.)

Recall that we introduced an order relation on the set of signed sets of an oriented matroid $\mathcal{O}$ by defining $X \leqq Y$ if and only if $X_+ \subset Y_+$ and $X_- \subset Y_-$ for $X, Y \in \mathcal{O}$. The lattice $(\mathcal{O}, \leqq)$ is called the *complex* of $\mathcal{O}$ ([10]).

To parallel the above embedding of adjoints into the extension lattice we shall prove that the complex $(\tilde{\mathcal{O}}^\perp, \leqq)$ where $\tilde{\mathcal{O}}$ is the adjoint of an oriented matroid $\mathcal{O}$ can be embedded into the extension lattice $E(\mathcal{O})$. To simplify matters we shall restrict ourselves to simple oriented matroids.

Theorem 3.2. *Let $\tilde{\mathcal{O}}$ be an adjoint of the simple oriented matroid $\mathcal{O}$ and let φ be given as in* (2.2). *Then the mapping*

$$f\colon \tilde{\mathcal{O}}^\perp \to E(\mathcal{O})$$

$$\tilde{X} \mapsto (\mathcal{Y}, \mathcal{X})$$

$$\textit{where} \quad \mathcal{Y} = \varphi(\tilde{X}_+) \cup -\varphi(\tilde{X}_-) \quad \textit{and} \quad \mathcal{X} = \varphi(\tilde{X}_0) \cup -\varphi(\tilde{X}_0)$$

defines an embedding of the complex $(\tilde{\mathcal{O}}^\perp, \leqq)$ into the extension lattice $E(\mathcal{O})$.

Note, that by the definition of f, the induced map f, mapping $\tilde{X}_0$ to the corresponding $\mathcal{X}$, is exactly the embedding of the lattice of flats of $\tilde{\mathcal{O}}$ into $E(\mathcal{O})$.

Proof of Theorem 3.2. To prove that f is injective we shall use the inverse of f defined as

$$g\colon E(\mathcal{O}) \to \tilde{\mathcal{O}}^\perp$$

$$(\mathcal{Y}, \mathcal{X}) \mapsto g(\mathcal{Y}, \mathcal{X})$$

where for all $\tilde{e} \in \tilde{E}$ (the underlying set of $\tilde{\mathcal{O}}$)

$$g(\mathcal{Y}, \mathcal{X})_{\tilde{e}} = \begin{cases} + & \text{if } \varphi(\tilde{e}) \in \mathcal{Y} \\ - & \text{if } \varphi(\tilde{e}) \in -\mathcal{Y} \\ 0 & \text{if } \varphi(\tilde{e}) \in \mathcal{X}. \end{cases}$$

If $\tilde{X} \in \tilde{\mathcal{O}}^\perp$ and $\tilde{Z} := g(f(\tilde{X}))$ then $\tilde{Z}_{\tilde{e}} = +$ if and only if $\varphi(\tilde{e}) \in \varphi(\tilde{X}_+) \cup -\varphi(\tilde{X}_-)$, i.e. if and only if $\varphi(\tilde{e}) = \varphi(\tilde{q})$ or $\varphi(\tilde{e}) = -\varphi(\tilde{q})$ for some $\tilde{q} \in \tilde{X}_+$ respec. $\tilde{q} \in \tilde{X}_-$. Because φ is bijective $\varphi(\tilde{e}) = -\varphi(\tilde{q})$ $(\tilde{q} \in \tilde{X}_-)$ can not occur and we have $Z_{\tilde{e}} = +$ if and only if $\tilde{e} \in \tilde{X}_+$. The "$-$" and "0" cases are similiar which proves $X = Z$ or f is injective. Clearly, f is order preserving and hence f is an embedding of $\tilde{\mathcal{O}}^\perp$ into $f(\tilde{\mathcal{O}}^\perp)$. To prove $f(\tilde{\mathcal{O}}^\perp) \subseteqq E(\mathcal{O})$ we shall use the following reduction steps.

Lemma 3.3. $f(\mathscr{C}(\tilde{\mathcal{O}}^\perp)) \subseteqq E(\mathcal{O})$.

Lemma 3.4. *$\tilde{X}^1, \tilde{X}^2 \in \tilde{\mathcal{O}}^\perp$ implies $f(\tilde{X}^1 \cdot \tilde{X}^2) \in E(\mathcal{O})$.*

Indeed, $E(\mathcal{O})$ is closed under taking products, i.e. $(\mathcal{Y}_1, \mathcal{X}_1) \in E(\mathcal{O})$ and $(\mathcal{Y}_2, \mathcal{X}_2) \in E(\mathcal{O})$ imply $(\mathcal{Y}_1 \cup (\mathcal{Y}_2 \cap \mathcal{X}_1), \mathcal{X}_1 \cap \mathcal{X}_2) \in E(\mathcal{O})$.

Geometrically, Lemma 3.4 states that, given two point extensions $\mathcal{O} \cup p_1$ and $\mathcal{O} \cup p_2$ of $\mathcal{O}$, there exists a point extension $\mathcal{O} \cup p_3$ such that, if all three point extensions could be carried out simultaneously, p_3 would ly in "general position" on the line segment $\overline{p_1p_2}$ nearby the point p_1.

Since every element $Z\in\mathcal{O}^\perp$ is the product of cocircuits $Y\in\mathscr{C}(\mathcal{O}^\perp)$ ([4]) Lemma 3.3 will then prove the theorem. To prove the lemmata we use a characterization of localizations which we shall introduce now.

Let $D:=C_0^1\cap C_0^2$ be a coline of an oriented matroid $\mathcal{O}$ defined by cocircuits $C^1, C^2\in\mathscr{C}(\mathcal{O}^\perp)$. Clearly $C^1\neq\pm C^2$ and using the circuit exchange axiom (eliminating all elements which separate C^1 and C^2 resp. C^2 and $-C^1$) we can construct a sequence $X^1, \dots, X^k$ of cocircuits of $\mathcal{O}$ containing C^1 and C^2 such that with $X^{k+1}:=-X^1$, for every $i=1, \dots, k$, X^i is *compatible* to X^{i+1}, i.e. X^i and X^{i+1} are not separated by any element $e\in E$. Such a sequence is unique up to cyclic shifting and reversal. Furthermore, all hyperplanes of $\mathcal{O}$ containing D are represented by some X_0^i $(i=1, \dots, k)$. We shall call such a sequence a *compatible bundle* of D. Note, that if $(X^1, \dots, X^k)$ is a compatible bundle, then $(X^2, \dots, X^k, -X^1)$ resp. $(-X^k, X^1, \dots, X^{k-1})$ are also compatible bundles. We shall say that these are obtained from $(X^1, \dots, X^k)$ by "cyclic shifting". With this definition we characterize localizations of $\mathcal{O}$ as follows:

Lemma 3.5. *Let $\mathcal{O}$ be an oriented matroid and $\mathscr{Y}, \mathscr{X}\subseteq\mathscr{C}(\mathcal{O}^\perp)$ such that $\mathscr{Y}\cup -\mathscr{Y}\cup\mathscr{X}$ is a partition of $\mathscr{C}(\mathcal{O}^\perp)$. Then $(\mathscr{Y}, \mathscr{X})$ is a localization of $\mathcal{O}$ if and only if*

(3.1) *for every coline D of $\mathcal{O}$ and every compatible bundle $X^1, \dots, X^k$ of D such that $X^1\in\mathscr{Y}$ there exists an index $i\in\{1, \dots, k\}$ such that $X^j\in\mathscr{Y}$ $(j<i)$ and $X^j\in-\mathscr{Y}$ $(j>i)$.*

Proof of Lemma 3.5. To prove the lemma we shall use the definition of localizations given by Las Vergnas ([9]), i.e.

(3.2) $\mathscr{Y}\cup-\mathscr{Y}\cup\mathscr{X}$ is a partition of $\mathscr{C}(\mathcal{O}^\perp)$.

(3.3) The set $\{Z_0|Z\in\mathscr{X}\}$ is a linear subclass, i.e. $X, Y\in\mathscr{X}$, $D:=X_0\cap Y_0$ a coline of $\mathcal{O}$ implies $Z\in\mathscr{X}$ for all $Z\in\mathscr{C}(\mathcal{O}^\perp)$ s.t. $D\subset Z_0$.

(3.4) For every coline D there is at least one element $e\in E\setminus D$ which has the same sign in every $Y\in\mathscr{Y}$, $D\subset Y_0$ which contains it (i.e. $\mathscr{Y}$ forms an "angle" in D). Moreover, all elements $e\in Y_0\setminus D$ for $Y\in\mathscr{X}$ and $D\subset Y_0$ have this property.

Assume that $(\mathscr{Y}, \mathscr{X})$ has the property (3.1) of Lemma 3.5. We will show that $(\mathscr{Y}, \mathscr{X})$ satisfies (3.2)—(3.4). To see that $\mathscr{X}$ (more precisely the set $\{Z_0|Z\in\mathscr{X}\}$) is a linear subclass, let D be a coline of $\mathcal{O}$ and let $(X^1, \dots, X^k)$ be a compatible bundle of D. It is easy to conclude from condition (3.1) — using cyclic shifting, if necessary — that either all the X^i's are in $\mathscr{X}$ or at most one of them (determined by the index i) is in $\mathscr{X}$. That is to say, $\mathscr{X}$ is a linear subclass.

Next we show that (3.4) is satisfied. Let D be a coline of $\mathcal{O}$ and let $(X^1, \dots, X^k)$ be a compatible bundle of D. If the X^j's are all in $\mathscr{X}$, $\mathscr{Y}$ forms trivially an angle in D. Otherwise, we assume $X^j\notin\mathscr{X}$ for some j. Shifting cyclically and reversing signs if necessary, we may assume that $X^1\in\mathscr{Y}$. By Lemma 3.4 if follows that $X^1, \dots, X^{i-1}\in\mathscr{Y}$ and $X^{i+1}, \dots, X^k\in-\mathscr{Y}$ for some $i\in\{1, \dots, k\}$. Since every X^j is compatible with X^{j+1}, it is easy to see that for every $e\in X_0^i\setminus D$ either $e\in X_+^i$ $(j<i)$ and $e\in X_-^i$ $(j>i)$ or vice versa. Now (3.4) follows immediately.

Assume now conversely that the pair $(\mathscr{Y}, \mathscr{Z})$ satisfies (3.2)—(3.4). To show (3.1), let D be a coline of $\mathcal{O}$ and let $(X^1, \ldots, X^k)$ be a compatible bundle such that $X^1 \in \mathscr{Y}$. By (3.4) there exists an $e \in E \setminus D$ that has the same sign in every $Y \in \mathscr{Y}$, $D \subset Y_0$. Let $i \in \{1, \ldots, k\}$ such that $e \in X_0^i$. Since the X^j's are compatible, either $e \in X_+^j$ $(j<i)$ and $e \in X_-^j$ $(j>i)$ or vice versa. Since e should have the same sign in every. $Y \in \mathscr{Y}$, we conclude that $X^j \in \mathscr{Y}$ $(j<i)$ and $X^j \in -\mathscr{Y}$ $(j>i)$, i.e. (3.1) holds. ∎

Proof of Lemma 3.3. Let $\tilde{X} \in \mathscr{C}(\tilde{\mathcal{O}}^\perp)$ be a cocircuit of $\tilde{\mathcal{O}}$, $f(\tilde{X}) = (\mathscr{Y}, \mathscr{Z})$ and $X^1, \ldots, X^k$ a compatible bundle of a coline D of $\mathcal{O}$ with $X^1 \in \mathscr{Y}$. Clearly $\mathscr{Z} = -\mathscr{Z}$ and since φ is bijective $\mathscr{Y} \cup -\mathscr{Y} \cup \mathscr{Z}$ is a partition of $\mathscr{C}(\mathcal{O}^\perp)$. Hence to prove $(\mathscr{Y}, \mathscr{Z}) \in E(\overline{\mathcal{O}})$ we can use Lemma 3.5 and verify the existence of an index r with $X^i \in \mathscr{Y}$ $(i<r)$ and $X^i \in -\mathscr{Y}$ $(i>r)$.

Since φ is surjective there exist signs $s_j \in \{+, -\}$ and elements $\tilde{q}_j \in \tilde{E}$ $(j = 1, \ldots, k)$ such that $\varphi(\tilde{q}_j) = s_j X^j$. Let us show that

$$(3.5) \qquad X^j \in \mathscr{Y} \text{ if and only if } \tilde{q}_j \in \tilde{X}_{s_j}$$

holds for all $j = 1, \ldots, k$. By definition $X^j \in \mathscr{Y}$ if and only if $s_j \varphi(\tilde{q}_j) \in \mathscr{Y} = \varphi(\tilde{X}_+) \cup \cup -\varphi(\tilde{X}_-)$, i.e. if and only if either $s_j = +$ and $\tilde{q}_j \in \tilde{X}_+$ or $s_j = -$ and $\tilde{q}_j \in \tilde{X}_-$. Thus $X^i \in \mathscr{Y}$ if and only if $\tilde{q}_j \in \tilde{X}_{s_j}$.

To prove the lemma we shall show that the index

$$(3.6) \qquad r := \max\{i \in \{1, \ldots, k\} \mid \tilde{q}_j \in \tilde{X}_{s_j} \ \forall j < i\}$$

has the property $\tilde{q}_i \in \tilde{X}_{s_i}$ $(i<r)$ and $\tilde{q}_i \in -\tilde{X}_{s_i}$ $(i>r)$. If $r=k$ then there is nothing to prove, hence assume $r<k$. To complete the proof we use the following property:

$$(3.7) \qquad \forall j \in \{2, \ldots, k-1\} \exists \tilde{C} \in \mathscr{C}(\tilde{\mathcal{O}}) \tilde{q}_1 \in \tilde{C}_{s_1}, \ \tilde{q}_j \in -\tilde{C}_{s_j},$$
$$\tilde{q}_{j+1} \in \tilde{C}_{s_{j+1}}, \ \underline{\tilde{C}} = \{\tilde{q}_1, \tilde{q}_j, \tilde{q}_{j+1}\}$$

Clearly, if (3.7) holds and $r<k$, then there is a $\tilde{C} \in \mathscr{C}(\tilde{\mathcal{O}})$ with $\tilde{q}_1 \in \tilde{C}_{s_1}$, $\tilde{q}_r \in -\tilde{C}_{s_r}$ and $\tilde{q}_{r+1} \in \tilde{C}_{s_{r+1}}$. Since $\underline{\tilde{C}} = \{\tilde{q}_1, \tilde{q}_r, \tilde{q}_{r+1}\}$ and $\tilde{C}$ is orthogonal to $\tilde{X}$ we obtain $\tilde{q}_{r+1} \in \tilde{X}_{-s_{r+1}}$ (namely $\tilde{C}$ and $\tilde{X}$ agree on signs at component 1 and r) or by (3.5) $X^{r+1} \in -\mathscr{Y}$. Repeating this argument for $r+1, r+2, \ldots, k$ proves $X^i \in -\mathscr{Y}$ $(i>r)$.

To prove (3.7) let $j \in \{2, \ldots, k-1\}$ and $p \in X_0^j \setminus D$. It is not hard to verify that $f(\psi(p)) = (\mathscr{Y}^p, \mathscr{Z}^p)$ is a localization of $\mathcal{O}$. Since $\varphi(\psi(p)_+) = \varphi(\{\tilde{e} \mid \tilde{e} \in \psi(p)_+\})$ we may use property (2.3) of φ and ψ to obtain $\varphi(\psi(p)_+) = \varphi(\{\tilde{e} \mid p \in \varphi(\tilde{e})_+\})$. Hence $Y \in \varphi(\psi(p)_+) \cup -\varphi(\psi(p)_-)$ if and only if $Y \in \operatorname{im} \varphi$ and $p \in Y_+$ or $-Y \in \operatorname{im} \varphi$ and $p \in Y_-$. And similarily $Z \in \mathscr{Z}$ if and only if $Z \in \mathscr{C}(\mathcal{O}^\perp)$ and $p \in Z_0$. Thus $(\mathscr{Y}^p, \mathscr{Z}^p)$ is a localization of $\mathcal{O}$ corresponding to the extension of $\mathcal{O}$ by a parallel element p'. Since $p \in X_0^j \setminus D$ and $j>1$, $p \in \underline{X}^1$, i.e. either $X^1 \in \mathscr{Y}^p$ or $X^1 \in -\mathscr{Y}^p$. Assume w.l.o.g. (take $f(-\psi(p))$ if necessary) $X^1 \in \mathscr{Y}^p$, then $p \in X^k$ because X^1 is compatible to $-X^k$ (and $j<k$, i.e. $p \notin X_0^k \setminus D$). Thus $X^k \in -\mathscr{Y}^p$ and we can use Lemma 3.4 to find an index j_0 with $X^i \in \mathscr{Y}^p$ $(i<j_0)$ and $X^i \in -\mathscr{Y}^p$ $(i>j)$. Since $p \in X_0^j \setminus D$ we have $X^j \in \mathscr{Z}^p$ which proves $j = j_0$.

Since $X^1 \in \mathscr{Y}^p$ or equivalently $p \in X^1_+$ respec. $p \in (s_1 X^1)_{s_1}$ we obtain $p \in \varphi(\tilde{q}_1)_{s_1}$ and by (2.3) $\tilde{q}_1 \in \psi(p)_{s_1}$. The same argument also yields $\tilde{q}_j \in \psi(p)_0$ and $\tilde{q}_{j+1} \in \in \psi(p)_{-s_{j+1}}$. Since $X^1, \dots, X^k$ is a compatible bundle of the coline D, i.e. $D \subseteqq X^i_0$ $(i=1, \dots, k)$ there is a line L of $\tilde{\mathcal{O}}$ which contains all points $\tilde{q}_i$ $(i=1, \dots, k)$. Hence there is a circuit $\tilde{C} \in \mathscr{C}(\tilde{\mathcal{O}})$ such that $\underline{\tilde{C}} = \{\tilde{q}_1, \tilde{q}_j, \tilde{q}_{j+1}\}$ and (w.l.o.g.) $\tilde{q}_1 \in \tilde{C}_{s_1}$. Since $\tilde{C}$ and $\psi(p)$ are orthogonal and $\underline{\tilde{C}} \cap \underline{\psi(p)} = \{\tilde{q}_1, \tilde{q}_{j+1}\}$, $\tilde{q}_1 \in \tilde{C}_{s_1}$ implies $\tilde{q}_{j+1} \in \in \tilde{C}_{s_{j+1}}$. To prove (3.7) it thus suffices to show $\tilde{q}_j \in -\tilde{C}_{s_j}$. Assume $\tilde{q}_j \in \tilde{C}_{s_j}$ and choose $p' \in X_0^{j+1} \setminus D$. The same arguments as above yield $\tilde{q}_1 \in \psi(p')_{s_1}$, $\tilde{q}_j \in \psi(p')_{s_j}$ and $\tilde{q}_{j+1} \in \psi(p')_0$, hence $\psi(p')$ and $\tilde{C}$ are not orthogonal, a contradiction. ∎

Proof of Lemma 3.4. Let $\tilde{X}^i \in \mathscr{C}(\mathcal{O}^\perp)$, $(\mathscr{Y}^i, \mathscr{Z}^i) := f(\tilde{X}^i)$ $(i=1,2)$ and define $\mathscr{Y}^3 := := \mathscr{Y}^1 \cup (\mathscr{Y}^2 \cap \mathscr{Z}^1)$, $\mathscr{Z}^3 := \mathscr{Z}^1 \cap \mathscr{Z}^2$ and $\tilde{X}^3 := g(\mathscr{Y}^3, \mathscr{Z}^3)$. It is very easy to show $\tilde{X}^3 = \tilde{X}^1 \cdot \tilde{X}^2$. Thus it remains to prove $(\mathscr{Y}^3, \mathscr{Z}^3) \in E(\mathcal{O})$. Clearly $\mathscr{Y}^3 \cup -\mathscr{Y}^3 \cup \mathscr{Z}^3$ is a partition of $\mathscr{C}(\mathcal{O}^\perp)$ and we can use Lemma 3.5 to prove $(\mathscr{Y}^3, \mathscr{Z}^3) \in E(\mathcal{O})$. Let D be a coline of $\mathcal{O}$ and $X^1, \dots, X^k$ a compatible bundle of D with $X^1 \in \mathscr{Y}^3$. If $X^1 \in \mathscr{Y}^1$ then by Lemma 3.5 there is an index i such that $X^j \in \mathscr{Y}^1 \subseteqq \mathscr{Y}^3$ $(j<i)$ and $X^j \in -\mathscr{Y}^1 \subseteqq \subseteqq \mathscr{Y}^3$ $(j>i)$. Thus assume $X^1 \notin \mathscr{Y}^1$, i.e. $X^1 \in \mathscr{Y}^2 \cap \mathscr{Z}^1$. By Lemma 3.5, we have $X^j \in \mathscr{Y}^2$ $(j<i)$ and $X^j \in -\mathscr{Y}^2$ $(j>i)$ for some index i. Hence, if $X^j \in \mathscr{Z}^1$ for all $j=1, \dots, k$, then $X^j \in \mathscr{Y}^3$ $(j<i)$ and $X^j \in -\mathscr{Y}^3$ $(j>i)$ for the same index i. Otherwise, $X^k \notin \mathscr{Z}^1$ (since $\mathscr{Z}^1$ is a linear subclass) and we may consider the compatible bundle $(X^k, \dots, X^1)$. The result follows immediately. ∎

Note that, given an adjoint $\tilde{\mathcal{O}}$ of $\mathcal{O}$, φ induces a bijection between the points of $\tilde{\mathcal{O}}$ and a representative system of cocircuits of $\mathcal{O}$, i.e. a set $\tilde{E} \subset \mathscr{C}(\mathcal{O}^\perp)$ such that a) for every $\tilde{e} \neq \tilde{f} \in \tilde{E}$, $\tilde{e} \neq -\tilde{f}$ and b) for every $Y \in \mathscr{C}(\mathcal{O}^\perp)$ there exist an $\tilde{e} \in \tilde{E}$ such that $\tilde{e} = \pm Y$. Thus, given such a system $\tilde{E} \subset \mathscr{C}(\mathcal{O}^\perp)$ and an oriented matroid $\tilde{\mathcal{O}}$ on $\tilde{E}$, we may ask, under which conditions $\tilde{\mathcal{O}}$ is an adjoint of $\mathcal{O}$. The following almost trivial result is in a sense the converse of Theorem 3.2.

Theorem 3.6. *Let $\tilde{E} \subset \mathscr{C}(\mathcal{O}^\perp)$ be a representative system of the cocircuits of $\mathcal{O}$ and let $\tilde{\mathcal{O}}$ be an oriented matroid on $\tilde{E}$ of the same rank as $\mathcal{O}$. Suppose that for every $\tilde{X} \in \tilde{\mathcal{O}}^\perp$, the pair $(\mathscr{Y}, \mathscr{Z}) := (\{\tilde{e} \mid \tilde{e} \in \tilde{E}$ and $\tilde{e} \in \tilde{X}_+$ or $-\tilde{e} \in \tilde{E}$ and $-\tilde{e} \in \tilde{X}_-\}, \{\tilde{e} \mid \tilde{e} \in \tilde{E}$ and $\tilde{e} \in \tilde{X}_0$ or $-\tilde{e} \in \tilde{E}$ and $-\tilde{e} \in \tilde{X}_0\})$ is a localization and that all localizations which correspond to extensions of $\mathcal{O}$ by parallel elements can be represented in this way by some $\tilde{X} \in \mathscr{C}(\mathcal{O}^\perp)$. Then $\tilde{\mathcal{O}}$ is an adjoint of $\mathcal{O}$.*

Proof. Let $e \in E$ be a point of $\mathcal{O}$ and let $(\mathscr{Y}_e, \mathscr{Z}_e)$ be the localization which corresponds to an extension of $\mathcal{O}$ by an element parallel to e, i.e. $\mathscr{Y}^e = \{Y \in \mathscr{C}(\mathcal{O}^\perp) \mid e \in Y^+\}$ and $\mathscr{Z}^e = \{Y \in \mathscr{C}(\mathcal{O}^\perp) \mid e \in Y_0\}$. By our assumption, there exists a (necessarily unique) element $\tilde{X}^e \in \mathscr{C}(\mathcal{O}^\perp)$ such that $\mathscr{Y}^e = \{\tilde{e} \mid \tilde{e} \in \tilde{E} \cap \tilde{X}^e_+$ or $-\tilde{e} \in \tilde{E} \cap \tilde{X}^e_-\}$ and $\mathscr{Z}^e = = \{\tilde{e} \mid \tilde{e} \in \tilde{E} \cap \tilde{X}^e_0$ or $-\tilde{e} \in \tilde{E} \cap \tilde{X}^e_0\}$. We define $\psi: E \to \mathscr{C}(\tilde{\mathcal{O}}^\perp)$ by setting $\psi(e) := \tilde{X}^e$ for every point $e \in E$. Let $\varphi: \tilde{E} \to \mathscr{C}(\mathcal{O}^\perp)$ be the natural injection. Then for every $e \in E$, $\tilde{e} \in \tilde{E}$ we have $e \in \tilde{e}_+ = \varphi(\tilde{e})_+$ iff $\tilde{e} \in \mathscr{Y}_e$ iff $\tilde{e} \in \tilde{X}^e_+ = \psi(e)^+$. Similarily, $e \in \varphi(\tilde{e})_-$ iff $\tilde{e} \in \psi(e)^-$. ∎

Remark. The additional assumption concerning parallel-extensions of $\mathcal{O}$ can be shown to be redundant. This shows that whether or not an oriented matroid does have an adjoint depends on the extension lattice only.

4. Some consequences

Provided we know that a given oriented matroid $\mathcal{O}$ has an adjoint, then Theorem 3.2 can be used to prove the existence of some extensions with special properties. The most obvious application is the one corresponding to Proposition 3.1:

Proposition 4.1. *If $\mathcal{O}$ has an adjoint, then any two flats of $\mathcal{O}$ can be intersected (here, "intersection" is understood to be defined in analogy to the unoriented case).*

Proof. A proof is easily derived, combining Theorem 3.2 and Proposition 3.1. ∎

This intersection problem has a nice interpretation, if we consider it from an "adjoint" point of view: Suppose $\mathcal{O}$ is the adjoint of some oriented matroid $\mathcal{O}'$, then every signed set $X \in \mathcal{O}$ corresponds to a point extension $\mathcal{O}' \cup p$ of $\mathcal{O}'$ and for every point e of $\mathcal{O}$, the sign of the e-th coordinate of X indicates the position of p relative to the hyperplane of $\mathcal{O}'$ which corresponds to e. Now we choose some hyperplane H_0' of $\mathcal{O}'$, corresponding to some point, say e_0 of $\mathcal{O}$ and declare it to be the "hyperplane at infinity". Then we consider only those point extensions $\mathcal{O}' \cup p$ of $\mathcal{O}'$ such that p lies on a prescribed side of H_0', i.e. we consider those signed sets $X \in \mathcal{O}$ such that, say $e_0 \in X_+$. This leads to the following

Definition ([10]). Let $\mathcal{O}$ be an oriented matroid on some finite set E and let $e_0 \in E$. Then $(\mathcal{O}, e_0) := \{X \in \mathcal{O} | e_0 \in X_+\}$ is an *affin matroid.*

(Note, however, that such a system is not a matroid at all!) Intuitively, one would define two flats F', G' of $\mathcal{O}'$ to be "parallel", (with respect to H_0') if for every point extension $\mathcal{O}' \cup p$, "p lies on F'" and "p lies on G'" imply "p lies on H_0'". If F and G are the flats of $\mathcal{O}$ corresponding to F' and G' (via the embedding function e), this amounts to say that for every $X \in \mathcal{O}$, $F \cup G \subset X_0$ implies $e_0 \in X_0$. This motivates the following

Definition ([10]). Let $(\mathcal{O}, e_0)$ be an affine matroid and let F, G be two flats of $\mathcal{O}$ such that $e_0 \notin F \cup G$. Then F and G are *parallel* if and only if $F \cup G \subset X_0$ implies $e_0 \in X_0$ for every $X \in \mathcal{O}$.

Euclid's theorem states, that in every affin space, given a hyperplane H' and a point $e' \notin H'$, there exists a hyperplane H'' parallel to H' which contains e'. Now, let H' be a hyperplane of $\mathcal{O}'$, different from H_0' and e' be a point of $\mathcal{O}'$ such that $e' \notin H'$. Then we may ask whether there exists a hyperplane H'' of $\mathcal{O}'$ parallel to H' and containing e'. "Translating" this problem to $\mathcal{O}$, the question is the following: Given a point $e \neq e_0$ of $\mathcal{O}$ (corresponding to H') and a hyperplane H of $\mathcal{O}$, such that $e \notin H$, does there exist a point p of $\mathcal{O}$ such that p is parallel to e and $p \in H$. Note that "p is parallel to e" means that p lies on the line spanned by e and e_0. Clearly, such a point p may not exist even if $\mathcal{O}$ is linear, and hence we have to postulate the existence of a point extension of $\mathcal{O}$ having this property:

Definition ([10]). Let $(\mathcal{O}, e_0)$ be an affine matroid. Then $(\mathcal{O}, e_0)$ is *euclidean,* if every hyperplane H of $\mathcal{O}$, $e_0 \notin H$, and every line containing e_0 can be intersected.

We call an oriented matroid *euclidean*, provided $(\mathcal{O}, e_0)$ is euclidean for every point e_0 of $\mathcal{O}$. Then Proposition 4.1 has as a corollary:

Proposition 4.2. *If $\mathcal{O}$ has an adjoint, $\mathcal{O}$ is euclidean.* ∎

Mandel ([10]) gives an example of an oriented matroid $\mathcal{O}$ such that $\underline{\mathcal{O}}$ is linear but $\mathcal{O}$ is not euclidean. It follows that the existence of an adjoint of $\underline{\mathcal{O}}$ does not imply the existence of an adjoint of $\mathcal{O}$.

We are grateful to the referee for many helpful suggestions and comments.

References

[1] L. J. BILLERA and B. S. MUNSON, *Polarity and inner products in oriented matroids,* Ithaca: Cornell University, School of Operations Research, (1981).

[2] R. E. BIXBY and C. COULLARD, Adjoints of binary matroids. *European Journal of Combinatorics, to appear.*

[3] R. G. BLAND, A combinatorial abstraction of linear programming. *Journal of Combinatorial Theory B* **23** (1977), 33—57.

[4] R. G. BLAND and M. LAS VERGNAS, Orientability of matroids, *Journal of Combinatorial Theory B* **24** (1978), 94—123.

[5] A. C. CHÉUNG, Adjoints of a geometry, *Canadian Math. Bul.* **17** (1974), 353—365.

[6] R. CORDOVIL, Oriented matroids of rank three and arrangements of pseudolines, Ann. Discr. Math. **17** (1983), 219—233.

[7] J. FOLKMAN and L. LAWRENCE, Oriented matroids, *Journal of Combinatorial Theory B* **25** (1975), 199—236.

[8] M. LAS VERGNAS, Extensions ponctuelles d'une géométric combinatoire orientée, *Acte du Coll Int CNRS*, **260**, (1976), 263—268.

[9] M. LAS VERGNAS, Convexity in oriented matroids, *Journal of Combinatorial Theory B* **29** (1980), 231—243.

[10] A. MANDEL, *Topology of oriented matroids.* University of Waterloo, Department of Combinatorics and Optimization (Dissertation), 1981.

[11] B. SPELLMAN-MUNSON, *Face lattices of oriented matroids.* Ithaca: Cornell University (Dissertation), 1981.

A. Bachem, W. Kern

Mathematisches Institut
Universität zu Köln
D—5000 Köln 41
West Germany